Wasserstein limits for empirical measures of Dirichlet diffusion processes

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Outline

- ♣ Reflecting diffusion processes on compact manifolds
- ♣ Killed diffusion processes on compact manifolds
- ♣ Diffusion processes on non-compact manifolds
- ♣ Semilinear SPDEs

Framework

- M: d-dimensional connected compact Riemannian manifold, possibly with a boundary ∂M and inward unit normal vector field N.
- $V \in C^2(M)$: $\mu(\mathrm{d}x) := \mathrm{e}^{V(x)}\mathrm{d}x$ is a probability measure on M.
- X_t: the (reflecting or killed) diffusion process generated by $L := \Delta + \nabla V$.
- $\mathcal{P}$: the set of probability measures on M.
- $\mathcal{P}_0 := \{\nu \in \mathcal{P} : \nu(\partial M) < 1\}$.

Framework

- Neumann (or closed, when $\partial M = \emptyset$) eigenproblem

$$Lu_m = -\theta_m u_m,\ m \ge 0,\ u_0 \equiv 1, \theta_0 = 0, Nu_m|_{\partial M} = 0.$$

- Dirichlet eigenproblem

$$L\phi_m = -\lambda_m \phi_m,\ m \ge 0,\ \phi_0 > 0, \lambda_0 > 0, \phi_m|_{\partial M} = 0.$$

Both $\{u_m\}_{m\ge0}$ and $\{\phi_m\}_{m\ge0}$ are ONBs of $L^2(\mu)$.

Since M is compact,

$$cm^{\frac{2}{d}} \le \theta_m,\ \lambda_m - \lambda_0 \le Cm^{\frac{2}{d}},\quad m \ge 0$$

holds for some constants $C > c > 0$.

Framework

Let X_t be the (reflecting if ∂M exists) diffusion process generated by $L := \Delta + \nabla V$. We investigate the limit of the empirical measure

$$\mu_t := \frac{1}{t}\int_0^t \delta_{X_s} \mathrm{d}s, \quad t > 0,$$

under the condition $t < \tau$ when the killed diffusion is concerned, where

$$\tau := \inf\{t \geq 0 : X_t \in \partial M\}.$$

Well known: for any initial distribution (not supported on ∂M when the killed diffusion is concerned), as $t \to \infty$ we have

- Ergodicity (LLN): $\mathbb{P}(\mu_t \to \mu \text{ weakly}) = 1$;
- Quasi-ergodicity (Conditional LLN):
  for any $\varepsilon > 0, f \in C_b(M)$,
  $\mathbb{P}(|\mu_t(f) - \mu_\infty(f)| \geq \varepsilon | t < \tau) \to 0.$

Aim

We investigate the convergence rate of μ_t under the Wasserstein distance $\mathbb{W}_2$ induced by the Riemannian metric ρ. In general, for $p \geq 1$,

$$\mathbb{W}_p(\mu_1,\mu_2) = \inf_{\pi\in\mathcal{C}(\mu_1,\mu_2)} \left(\int_{M\times M} \rho(x,y)^p \pi(\mathrm{d}x,\mathrm{d}y)\right)^{\frac{1}{p}},$$

where $\mathcal{C}(\mu_1,\mu_2)$ is the set of all couplings of μ_1 and μ_2.

We consider the convergence of μ_t in the following senses:

- $\mathbb{E}\mathbb{W}_2(\mu_t,\mu)^2 \to 0$ for the diffusion with reflecting or free boundary;
- $\mathbb{E}\big(\mathbb{W}_2(\mu_t,\mu_\infty)^2\big|t<\tau\big) \to 0$ for the killed diffusion;
- $\mathbb{W}_2\big(\mathbb{E}(\mu_t|t<\tau),\mu_\infty\big)^2 \to 0$ for the killed diffusion.

Reflecting or free boundary: compact manifold

Theorem (W./Jiexiang Zhu, aXiv:1906.03422)

(1) *When $\partial M = \emptyset$ or convex, uniformly in $x \in M$:*

$$\lim_{t\to\infty} \Big\{ t\mathbb{E}^x[\mathbb{W}_2(\mu_t, \mu)^2] \Big\} = \sum_{i=1}^\infty \frac{2}{\theta_i^2}.$$

In general,

$$\limsup_{t\to\infty} \sup_{x\in M} \Big\{ t\mathbb{E}^x[\mathbb{W}_2(\mu_t, \mu)^2] \Big\} \le \sum_{i=1}^\infty \frac{2}{\theta_i^2},$$

and there exists a constant $c > 0$ such that

$$\liminf_{t\to\infty} \inf_{x\in M} \Big\{ t\mathbb{E}^x[\mathbb{W}_2(\mu_t, \mu)^2] \Big\} \ge c \sum_{i=1}^\infty \frac{2}{\theta_i^2}.$$

The limit is finite if and only if $d \le 3$ since $\theta_i \sim i^{\frac{2}{d}}$

Reflecting or free boundary: compact manifold

(2) Let $d = 4$. There exists a constant $c > 0$ such that

$$\sup_{x\in M} \mathbb{E}^x[\mathbb{W}_2(\mu_t, \mu)^2] \le ct^{-1}\log(1+t), \quad t \ge 0.$$

If $M = \mathbb{T}^4$ and $V = 0$ then there exists a constant $c' > 0$ such that

$$\inf_{x\in M} \mathbb{E}^x[\mathbb{W}_2(\mu_t, \mu)^2] \ge \inf_{x\in M} \mathbb{E}^x[\mathbb{W}_1(\mu_t, \mu)^2] \ge c't^{-1}\log(1+t), \quad t \ge 0.$$

The proof for $M = \mathbb{T}^4$ and $V = 0$ relies on specific formula for eigenfunctions and Fourier transform of the volume measure. However, this argument does not work for other situations.

(3) Let $d \geq 5$. There exist constants $c \geq c' > 0$ such that

$$c' t^{-\frac{2}{d-2}} \leq \inf_{x\in M} \left\{\mathbb{E}^x[\mathbb{W}_1(\mu_t, \mu)]\right\}^2 \leq \sup_{x\in M} \mathbb{E}^x[\mathbb{W}_2(\mu_t, \mu)^2] \leq c t^{-\frac{2}{d-2}}.$$

$\mu_t^\nu := \mathbb{E}^\nu(\mu_t|t<\tau)$: compact manifold

Let $\partial M \neq \emptyset$. For any

$$\nu \in \mathcal{P}_0 := \{\nu \in \mathcal{P} : \nu(\partial M) < 1\},$$

we consider the conditional empirical measure

$$\mu_t^\nu := \mathbb{E}^\nu(\mu_t|t<\tau), \quad t>0,$$

where $\mu_t := \frac{1}{t}\int_0^t \delta_{X_s}\mathrm{d}s$.

Let $\{\phi_m, \lambda_m\}_{m\geq 0}$ be the Dirichlet eigenfunctions/eigenvalues of $-L$.

Let $\mu_\infty = \phi_0^2\mu$.

$\mu_t^\nu := \mathbb{E}^\nu(\mu_t|t<\tau)$: compact manifold

Theorem (W. JFA21')

For any $\nu \in \mathcal{P}_0$,

$$\lim_{t\to\infty}\{t^2\mathbb{W}_2(\mu_t^\nu,\mu_\infty)^2\} = \frac{1}{\{\mu(\phi_0)\nu(\phi_0)\}^2}\sum_{m=1}^\infty \frac{\{\nu(\phi_0)\mu(\phi_m)+\mu(\phi_0)\nu(\phi_m)\}^2}{(\lambda_m-\lambda_0)^3} > 0,$$

the limit is finite if either $d\le 5$, *or* $d\ge 6$ *and* $\nu = h\mu$ *with* $h\in L^p(\mu)$ *for some* $p>\frac{2d}{d+6}$.

The convergence of $\mathbb{W}_2(\mu_t^\nu,\mu_\infty)^2$ is of order t^{-2}, which is faster than t^{-1} for $\mathbb{E}^\nu\mathbb{W}_2(\mu_t,\mu)^2$.
Recall $\mu_t^\nu = \mathbb{E}^\nu(\mu_t|t<\tau)$.
What about $\mathbb{E}^\nu[\mathbb{W}_2(\mu_t,\mu_\infty)^2|t<\tau]$?

$\mathbb{E}^\nu(\mathbb{W}_2(\mu_t,\mu_\infty)^2|t<\tau)$: compact manifold

Theorem (W. JEMS revision)

(1) *There exists* $c\in(0,1]$ *with* $c=1$ *when* ∂M *is convex, such that*

$$c\sum_{m=1}^\infty \frac{2}{(\lambda_m-\lambda_0)^2}$$

$$\le \liminf_{t\to\infty}\inf_{T\ge t}\Big\{t\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t,\mu_\infty)^2\big|T<\tau\big]\Big\}$$

$$\le \limsup_{t\to\infty}\sup_{T\ge t}\Big\{t\mathbb{E}^\nu\big[\mathbb{W}_2(\mu_t,\mu_\infty)^2\big|T<\tau\big]\Big\}$$

$$\le \sum_{m=1}^\infty \frac{2}{(\lambda_m-\lambda_0)^2}.$$

Note: $\sum_{m=1}^\infty \frac{2}{(\lambda_m-\lambda_0)^2}<\infty$ if and only if $d\le 3$

$\mathbb{E}^\nu(\mathbb{W}_2(\mu_t,\mu_\infty)^2|t<\tau)$: compact manifold

(2) When $d=4$, there exist constants $c_1, c_2>0$ such that

$$\sup_{T\ge t}\mathbb{E}^\nu\Big[\mathbb{W}_2(\mu_t,\mu_\infty)^2\Big|T<\tau\Big]\le ct^{-1}\log(t+1),\quad t\ge 1.$$

(3) When $d\ge 5$, there exist constants $c_1, c_2>0$ such that

$$c_1 t^{-\frac{2}{d-2}}\le \inf_{T\ge t}\mathbb{E}^\nu\Big[\mathbb{W}_2(\mu_t,\mu_\infty)^2\Big|T<\tau\Big]$$

$$\le \sup_{T\ge t}\mathbb{E}^\nu\Big[\mathbb{W}_2(\mu_t,\mu_\infty)^2\Big|T<\tau\Big]\le c_2 t^{-\frac{2}{d-2}},\quad t\ge 1.$$

Non-compact manifold

Assume that M is non-compact possibly with a boundary ∂M which is convex outside a compact domain.

Let $p_t(x,y)$ be the heat kernel of the (Neumann) Markov semigroup P_t generated by L. Assume

$$\gamma(t) := \int_M (p_t(x,x) - 1)\mu(\mathrm{d}x) < \infty, \quad t > 0.$$

Since $\gamma(t)$ is deceasing in t, we have

$$\beta(\varepsilon) := \int_\varepsilon^1 \mathrm{d}s \int_s^1 \gamma(t)\mathrm{d}t < \infty, \quad \varepsilon \in (0,1].$$

Moreover, assume that for any $\varepsilon > 0$

$$\alpha(\varepsilon) := \int_M \mathbb{E}^x \rho(x, X_\varepsilon)^2 \mu(\mathrm{d}x)$$
$$= \int_M \rho(x,y)^2 p_\varepsilon(x,y)\mu(\mathrm{d}x)\mu(\mathrm{d}y) < \infty.$$

Non-compact manifold

Theorem (W. SPA revision)

Under the above assumptions we have

(1) *If the initial distribution* ν *satisfies* $\nu \le c_0\mu$ *for some constant* $c_0 > 0$*, then there exists a constant* $c > 0$ *such that*

$$\mathbb{E}^{\nu_0}\mathbb{W}_2(\mu_t, \mu)^2 \le c \inf_{\varepsilon\in(0,1]} \big\{\alpha(\varepsilon) + t^{-1}\beta(\varepsilon)\big\}, \quad t \ge 1.$$

(2) *If* P_t *is ultracontravtive, then there exists a constant* $c > 0$ *such that for any* $x \in M$*,*

$$\mathbb{E}^x\mathbb{W}_2(\mu_t, \mu)^2 \le c\Big[t^{-1} \sup_{s\in[0,1]} \mathbb{E}^x|X_s|^2 + \inf_{\varepsilon\in(0,1]} \big\{\alpha(\varepsilon) + t^{-1}\beta(\varepsilon)\big\}\Big], \quad t \ge 1.$$

Non-compact manifold: lower bound

(3) For any initial distribution, there exists a constant $c > 0$ such that

$$\mathbb{E}\mathbb{W}_1(\mu_t, \mu) \geq ct^{-\frac{1}{2}}, \quad t \geq 1.$$

(4) Let $d \geq 5$, $\mu(|\nabla V|) < \infty$ and there exists a constant $K \geq 0$ such that

$$\text{Ric} \geq -K, \quad -\text{Hess}_V \geq K, \quad V \leq K.$$

Then for any probability measure ν, there exists a constant $c(\nu) > 0$ such that

$$\mathbb{E}^\nu W_1(\mu_t, \mu) \geq c(\nu)t^{-\frac{1}{d-2}}, \quad t \geq 1.$$

Non-compact manifold: example

Let $M = \mathbb{R}^d$ and $V(x) = -\kappa|x|^p + W(x)$ for some constants $\kappa > 0, p > 1$, and some function $W \in C^1(M)$ with $\|\nabla W\|_\infty < \infty$. Then for any initial distribution ν with $\nu \le c_0\mu$ for some constant $c_0 > 0$, there exists a constant $c > 0$ such that for any $t \ge 1$,

$$\mathbb{E}^\nu \mathbb{W}_2(\mu_t, \mu)^2 \le \begin{cases} ct^{-\frac{2(p-1)}{(d-2)p+2}}, & \text{if } 4(p-1) < dp, \\ ct^{-1}\log(1+t), & \text{if } 4(p-1) = dp, \\ ct^{-1}, & \text{if } 4(p-1) > dp. \end{cases}$$

and

$$\left\{\mathbb{E}^\nu W_1(\mu_t, \mu)\right\}^2 \ge c' t^{-\frac{2}{(d-2)\vee 2}}, \quad t \ge 1.$$

The order t^{-1} is exact for $4(p-1) > dp$, but for larger d, the upper bound and lower bound have the same order only when $p \to \infty$.

SPDEs [W. 2102.00361]

Consider the following SDE on a separable Hilbert space $\mathbb{H}$:

$$\mathrm{d}X_t = \{\nabla V(X_t) - AX_t\}\mathrm{d}t + \sqrt{2}\,\mathrm{d}W_t,$$

where W_t is the cylindrical Brownian motion on $\mathbb{H}$, i.e.

$$W_t = \sum_{i=1}^\infty B_t^i e_i, \quad t \geq 0$$

for an orthonormal basis $\{e_i\}_{i\geq 1}$ of $\mathbb{H}$ and a sequence of independent one-dimensional Brownian motions $\{B_t^i\}_{i\geq 1}$, $V \in C^1(\mathbb{H})$, and $(A, \mathcal{D}(A))$ is a positive definite self-adjoint operator with discrete spectrum $0 < \lambda_i \uparrow \infty$.

An example [W. 2102.00361]

Example. Let $\lambda_i \geq c_0 i^p$ for some constant $c_0 > 0$ and $p > 1$, ∇V be Lipschitz continuous and

$$|V(x)| \leq c(1 + |x|), \quad x \in \mathbb{H}$$

holds for some constant $c > 0$. Then there exists a a constant $\kappa > 0$ such that

$$\mathbb{E}^\mu[\mathbb{W}_2(\mu_t, \mu)^2] \leq \kappa(\log t)^{p^{-1}-1}, \quad t \geq 2.$$

Papers

1. W. J.-X. Zhu, Limit theorems in Wasserstein distance for empirical measures of diffusion processes on Riemannian manifolds, aXiv:1906.03422.
2. W., Precise limit in Wasserstein distance for conditional empirical measures of Dirichlet diffusion processes, JFA21'
3. W., Convergence in Wasserstein distance for empirical measures of Dirichlet diffusion processes on manifolds, JEMS revision arXiv:2005.09290.
4. W., Wasserstein convergence rate for empirical measures on noncompact manifolds, SPA revision arXiv:2007.14667.
5. W., Convergence in Wasserstein distance for empirical measures of semilinear SPDEs, arXiv:2102.00361.

Thank You